Research Article https://doi.org/10.12973/eu-jer.12.1.421

European Journal of Educational Research
Volume 12, Issue 1, 421 - 433.
ISSN: 2165-8714
http://www.eu-jer.com/

Generalization of Patterns Drawing of High-Performance Students Based on Action, Process, Object, and Schema Theory

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Received: August 21, 2022 ▪ Revised: December 1, 2022 ▪ Accepted: January 9, 2023

Abstract: This study is qualitative with descriptive and aims to determine the process of generalizing the pattern image of high performance students based on the action, process, object, and schema (APOS) theory. The participants in this study were high performance eighth-grade Indonesian junior high school. Assignments and examinations to gauge mathematical aptitude and interviews were used to collect data for the study. The stages of qualitative analysis include data reduction, data presentation, and generating conclusions. This study showed that when given a sequence using a pattern drawing, the subjects used a number sequence pattern to calculate the value of the next term. Students in the action stage interiorize and coordinate by collecting prints from each sequence of numbers in the process stage. After that, they do a reversal so that at the object stage, students do encapsulation, then decapsulate by evaluating the patterns observed and validating the number series patterns they find. Students explain the generalization quality of number sequence patterns at the schema stage by connecting activities, processes, and objects from one concept to actions, processes, and things from other ideas. In addition, students carry out thematization at the schematic stage by connecting existing pattern drawing concepts with general sequences. From these results, it is recommended to improve the problem-solving skill in mathematical pattern problems based on problem-solving by high performance students', such as worksheets for students.

Keywords: *APOS, generalization, high-performance, pattern drawing.*

To cite this article: Firdaus, A. M., Murtafiah, W., Lukitasari, M., Lestari, N. D. S., Ernawati, T., & Widodo, S. A. (2023). Generalization of patterns drawing of high-performance students based on action, process, object, and schema theory. *European Journal of Educational Research, 12*(1), 421-433. https://doi.org/10.12973/eu-jer.12.1.421

Introduction

Mathematics taught in schools in Indonesia consists of material that has been standardized by the Educational Assessment and Curriculum Standards Board by establishing learning outcomes in the Independent Curriculum or Core Competencies and Basic Competencies in the 2013 Curriculum (Astuti & Anwar, 2018; Situmorang et al., 2015; Suciati et al., 2020). Even though mathematics material has been standardized, schools can determine indicators of competency achievement and Learning Objectives Achievement Criteria (Bakar, 2018; Glaesser, 2019; Madani, 2019; Rafiola et al., 2020; Syarifuddin, 2018). Because indicators of competency achievement and learning achievement criteria are different for each school, the levels of depth of mathematics material given to students may also be other (Diana et al., 2020; Mazana et al., 2019). But in general, these differences are not visible because learning outcomes or core competencies and essential competencies for each school are the same (Diana et al., 2020; Mazana et al., 2019).

Mathematics is described as the study of patterns (Samson, 2011; Tikekar, 2009). This pattern can be seen in the form of math problems which can be presented in various forms, for example, numerical, sequence, narrative, or contextual examples (Hodnik Čadež & Manfreda Kolar, 2015; Morrison et al., 2015; Muhtarom et al., 2019; Samson, 2011). Mathematical patterns include the act of counting, comparing, classifying, measuring, estimating, and symbolizing, and this process makes students' mathematical abilities and knowledge in schools meaningful (Aunio & Räsänen, 2016; Fernández Cueli et al., 2020; Rosa et al., 2016; Rosa & Orey, 2016). Mathematical patterns can form an arrangement of

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several numbers or colors that can create specific rules (Fox, 2005, 2006; McClelland & Rumelhart, 1985; Nern et al., 2015). for example, on a calendar, there is an arrangement of numbers either horizontally, descending, or diagonally; On paving coloring, tiles can also be garden arrangements that form a pattern. Mathematics learning, especially those related to pattern drawing, has been taught to children from kindergarten to junior high school. Students learn patterns drawing, such as repeating patterns and developing patterns (Björklund & Pramling, 2014; Lee & Freiman, 2006; Radford, 2008a, 2008b; Warren & Cooper, 2007). Learning about patterns at the childhood level still uses props or media such as balls, paper, blocks, and other concrete objects. But at secondary school, higher pattern learning begins at the junior high school level.

Activities related to pattern drawing are essential to realizing the relationship between mathematics, understanding systems, and mathematical logic (Arseven, 2015; Clements & Sarama, 2014; Lave, 1990; Singer et al., 2016). Mathematical patterns are represented by regularities which may be numerical, spatial, or logical networks (Hawes et al., 2019; Mulligan & Mitchelmore, 2009; Wang et al., 2015). The results of previous studies show that when presented with a number pattern, each student uses a unique technique to generalize the pattern (Tikekar, 2009). For this reason, learning mathematical patterns, in theory, can develop students' cognitive abilities, such as counting skills, reasoning, communication, associations, and problem-solving (Aunio & Räsänen, 2016; Iuculano et al., 2014; Montague-Smith et al., 2017; Xu et al., 2022). Some studies even mention that psychomotor ability, such as the ability to put in order and structure students' thinking strategies, can also be developed by studying mathematical patterns (Biggs & Collis, 2014; Blanton et al., 2015). This research shows that learning the pattern of numbers is essential to enrich students' cognitive structures.

In addition to studying these mathematical patterns, the study results show that generalizing a mathematical pattern can improve students' algebraic thinking and construct concepts of variables and functions (Lee & Freiman, 2004). Besides that, it also helps students to understand symbolic representations and their relationship to prior arithmetic knowledge (Lannin, 2005; Lannin et al., 2006). Therefore, the generalization of mathematical patterns can facilitate students to develop from thinking arithmetically to formal algebra. Solving problems related to mathematical patterns allows individuals to observe and generalize themselves and translate them symbolically (Radford, 2008a). Finding patterns is a fundamental step for making generalizations and, at the same time, an event for approaching algebra (Johnston-Wilder & Mason, 2005; Radford & Peirce, 2006; Smith, 2017; Zazkis & Liljedahl, 2002a, 2002b). This shows that studying the generalization of the pattern of numbers is essential to enrich students' cognitive structures. Although research states that to increase students' cognitive structures, teachers need to help students develop mental representations, relate them to representations already stored in memory and recall them when needed (Richland & Simms, 2015). Even though the results of this study are contrary to research in general which states that to develop students' cognitive structures, they must form their patterns and relationships or connection ability (Blake & Pope, 2015; Kusumadewi et al., 2019).

Mathematics learning activities related to number patterns and their generalizations indicate that these two things need to be given to students so that they can be used to improve their cognitive structure. For example, in a previous study, students were asked to solve the problem "There are seven boxes of cakes. Each box contains three Cakes. How many cakes in total?..." (Mutaqin, 2017). To solve this problem, students who are used as research subjects solve them with tools such as drawing 7 boxes containing 3 small circles in each box, then counting one by one (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21), Some say threes (3, 6, 9, 12, 15, 18, 21) so that the answer is the number of small circles in the whole box is 21 (see Figure 1). Still, other subjects familiar with multiplication answered the question with $7x3 = 21$.

Figure 1. Student Answers on Pattern Problems

For students, building a pattern with more powerful words seems more complicated than determining the closest term from the existing pattern. This shows that they can find the nearest term from the pattern drawing through counting strategies (build/create a dwarf to explain the situation and measure the desired object) or recursive procedures (explain the nexus with the same discriminator as the previous term), rather than seeking generalizations from the term pattern the greater one. As a result, researchers are interested in the prospect of conducting research in Makassar.

Researchers are interested in researching whether the results of generalizing image patterns in other countries will produce results similar to the research in Makassar. This fact contradicts previous studies, which stated that mathematics always talks about generalization patterns and their relationships (Reys et al., 2020; Samson, 2011; Tikekar, 2009). Mathematics will enable students to organize, analyze and synthesize existing information (Reys et al., 2020). If students already have these three abilities, they can be used to solve real-life situations. This is in line with previous studies, which state that students are involved in the investigative process to solve problems: (1) students look for patterns in stories; (2) students recognize patterns and describe patterns using different methods, and (3) students generalize patterns and relate them to stories (Herbert & Brown, 1997). That is, investigating patterns, recognizing patterns, and then generalizing patterns can be done with a problem-solving approach. Based on this explanation, this study aims to determine the process of generalizing the pattern image of high performance students based on the action, process, object, and schema (APOS) theory. This theory is used to study how individuals learn mathematical concepts, especially in the number pattern (Arnon et al., 2014a; Font Moll et al., 2016; Ndlovu & Brijlall, 2015).

Methodology

Research Design

This qualitative study uses descriptive data. It is the study of a symptom, event, fact, or occurrence to create a clear and detailed image of the outcomes of research student activities (Creswell, 2012; Fraenkel et al., 2012). Qualitative research was chosen because this study aims to determine the process of generalizing achievement student patterns based on the APOS.

Sample and Data Collection

The number of students in class VIII is 36 people. Demographically it consists of 11 male students and 25 female students. When viewed from the age, they are 13-14 years old.

All grade VIII students were given a mathematics ability test. Based on the mathematical ability test results, one student with high performance was selected, and the student was willing to be interviewed. The sampling technique used in this research is purposive sampling because the selection is not random. In addition, the sample in this study was adjusted to the research objectives, namely determining the process of generalizing the pattern image of high performance students based on the APOS theory.

The following instruments were used to gather data in this study: a) the researcher as the primary instrument, b) supporting instruments, namely (1) the mathematical ability test, and (2) The job of generalizing supplied patterns. The researcher conducted interviews directly with the research subject, which resulted in the researcher being used as the main instrument. Interviews were conducted by researchers in an unstructured manner so that interviews conducted on subjects followed the results of student work that had been written on the answer sheets. The interview items for each APOS stage can be seen in Table 1 (Arnon et al., 2014a).

Table 1. Interview Items For Each APOS Stage

APOS mental structure	***APOS mental mechanism***
Action • Determine the value of the next term if given a sequence using a pattern!	Interiorization: • How would you identify those on the assignment? • How do you understand a sequence and pattern shape?
Process • How do you determine the value of the next term? • Explain the differences in a sequence by observing the patterns of several tribes! • Explain and reflect on the transformation steps procedurally!	Coordination: • Determine the pattern of each row! • How do you coordinate information on tasks in designing strategy? • How will you coordinate the scores for the next term? Reversals: • How do you sequence/compile previous knowledge related to the problem?

Table 1. Continued

APOS mental structure	*APOS mental mechanism*
Object • Can you show that the pattern has specific characteristics and characteristics? Explain! • Is there a connection between this problem and the problem you understand? Explain! • How do you conceptually determine the value of the next term?	Encapsulation: • How do you determine the first term? • How do you choose the difference between each tribe? • How do you determine the formula for U_n? De-Encapsulation: • What is your purpose in checking the answer to the first term? • What is your goal in reviewing the different solutions for each tribe? • What is your purpose in checking the answer to the formula for the U_n?
Schema • Explain the generalization properties of patterns by connecting a concept's actions, processes, and objects with other ideas!	Thematization: • How do you relate the pattern concept to sequences in general?

A mathematical ability test as a supporting instrument is used to assess the mathematical competence of class VIII students. This test sheet consists of 5 questions in the form of descriptions of the number sequence material. The questions about the given sequences can be seen in Figure 2. Besides being used as the basis for determining which students are used as research subjects, this math ability test is also used by teachers to carry out daily assessments.

Table 3. Mathematical Ability Test of Sequence

No	Mathematical Problem
1	Determine U_n from the sequence of numbers 5, 9, 13, 17, ...
2	Determine the following two terms of the number sequence 3, 8, 15, 24, 35,...
3	Determine the following two terms of the number sequence 2, 4, 6, 10, 16, 26,...
4	Determine the following two terms of the number sequence 1, 3, 6, 10, 15,...
5	Determine U_{50} from the number sequence 5, 8, 11, 14, ...

Besides that, other supporting instruments, namely the task sheet generalizing a given pattern drawing. This instrument is used to characterize the research subject in generalizing mathematical patterns based on the APOS theory. This task sheet instrument consists of 3 questions related to pattern drawing (see Figure 3).

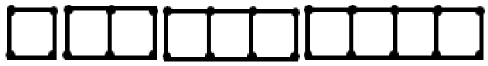
Data Collection

This research collects data through assignments and interviews. The subjects were instructed to perform the job within a specific time frame. Additionally, the researchers conducted in-depth interviews with the research subjects based on their replies to the supplied pattern generalization test. The interview results are captured using a handheld camera, referred to as the think-out-loud approach.

Analyzing of Data

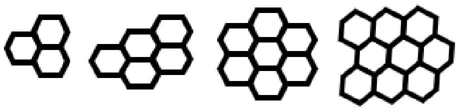
The term "data analysis" refers to the steps of qualitative data analysis in this study, namely data reduction, data presentation, and conclusion drafting (Firdaus et al., 2019; Juniati & Budayasa, 2017; Miles et al., 2018). The task of generalizing the drawing pattern can be seen in Figure 2.

1. Given the sequence as in the picture below.



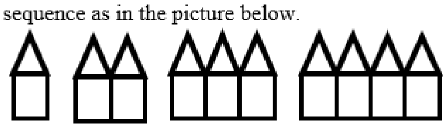
a) How many matchsticks are needed to make the box in Figure 6?
b) How many matchsticks are needed to make the box in Figure 7?
c) How many matchsticks are needed to make the box in Figure 20?

2. Given the sequence as in the picture below.



a) How to determine the number of sticks to make the miniature in the 5th picture?
b) How to determine the number of sticks to make the miniature in the 8th picture?
c) How to determine the number of sticks to make the miniature in the 33rd picture?

3. Given the sequence as in the picture below.



a. Mention the sequence's characteristics from the picture above!
b. Mention the general pattern relationship of the above sequence!
c. Find the value of the following term from the sequence based on your understanding!

Figure 2. The task of Drawing Pattern

The data analysis process begins with the reduction of test and interview data. APOS theory is used to analyze data in generalizing number patterns. Table 3 summarizes the indicators of the process of number pattern generalization (Arnon et al., 2014a).

Table 3. Indicators of the Number Pattern Generalization Process

The Step of APOS	Descriptor Theory
Action	• Find the next word using a number pattern if given a sequence.
Process	• Describe the procedure for determining the next word in a numerical pattern. • Demonstrate the distinction between sequences by identifying the pattern. • Procedurally explain the transformation stages.
Object	• Indicates the uniqueness of the number patterns. • Justify the relevance of this problem to the preceding one. • Decide the next term's conceptual meaning.
Schema	• Demonstrate the qualities of pattern generalization by associating a concept's activities, processes, and objects with those of other ideas.
Process	• Find the next word using a number pattern if given a sequence.

This research involves humans as subjects. For this reason, ethical clearance is needed to serve as a reference for the research team to uphold the values of integrity, honesty, and fairness in conducting research. Ethical approval was obtained from the research ethics committee at the Universitas Muhammadiyah Prof. Dr. Hamka Jakarta, Indonesia, with letter number 141/F.03.01/2022 dated 10 June 2022. This ethical clearance letter is valid until 10 June 2023.

Additionally, it is justified and validated to guarantee the credibility of the data. The credibility of the data in this study was obtained through technique triangulation. Technical triangulation, namely testing the validity of the data, is carried out on the same source but with different techniques (Creswell, 2012; Fraenkel et al., 2012).

Findings / Results

Following the findings of the students' work in Figure 3, the researchers conducted interviews with students to learn how they accomplished the action and process stages. According to researcher interviews about the results of the generalization task for students' drawing patterns, students determine the value of the term in a row using formulas and the manual method, namely drawing by selecting the difference in each sequence, adding the difference to the previous series, and finding the following term using the formula $U_n = a + (n-1)b$.

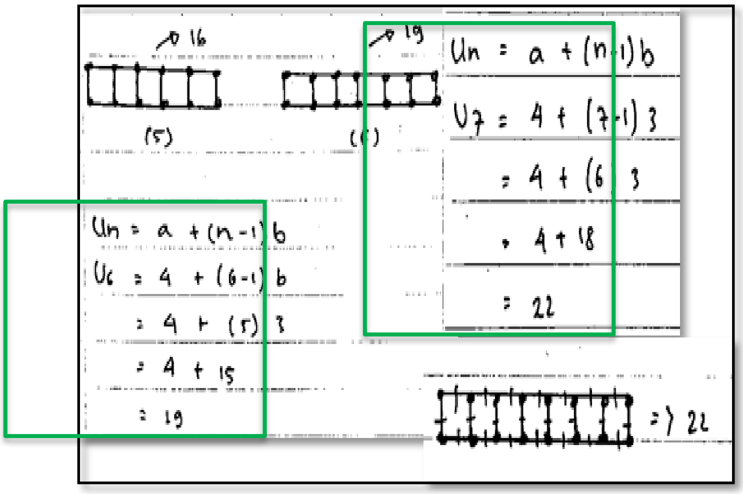


Figure 3. Student Work Results of Action Stage

According to the description above, students at the action stage determine the value of the next term by calculating the difference between successive terms based on the sequence, continue with adding the difference to the final term, and finally calculate the next term's value by repeating the previous method, namely adding the differentiator to the last term. This can be seen when interviewing the subject, who said that the number of squares in the picture makes up a regular pattern, namely 4 in Figure 3, 7 in the second picture pattern, 10 in the third picture, and 13 in the fourth picture pattern and has the same difference in each picture, namely 3 subjects. Initially used, pictures in determining the number of matchsticks, but the subject felt the number was too large, so the subject used a formula to determine the number of matchsticks for larger tribes. Finally, students determine the next term in the picture by applying a formula from prior experience, precisely the nth term formula or $U_n = a + (n-1)b$, and then utilizing a procedural technique to confirm the answers discovered.

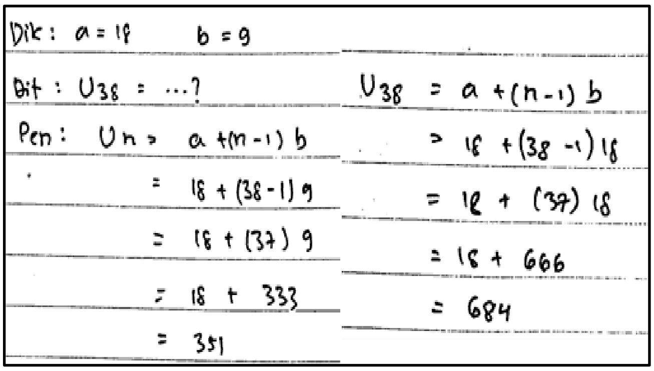


Figure 4. Student Work Results of Process Stage

Students describe how to identify the following term procedurally and visually by adding the difference between each phrase and utilizing a formula in the process stage based on Figure 4. Then, students compare or differentiate a series by focusing on the pattern and the difference between the terms in the sequence of created numbers, precisely the shape of the pattern drawing, the difference between two adjacent terms, and the total number of general terms. Additionally, students reflect on the transformation phases by identifying the nth term by comparing the results obtained through manual methods and formulae, first manually and then ensuring that the results obtained using correct procedures, and vice versa. This can be seen during the interview with the subject, who said that the subject used a formula to make it easier to find the answer, namely 351. When the difference is 2 times from the previous question, the procedure is replaced $Un = Un' + (n-1)\, b$ and then checked with the last general formula. After the results were obtained and checked using the formula $U_n = a + (n-1)\, b$, it was proven that the results were the same.

According to the description above, students interiorize the action at the process stage by describing the way of determining the value of the next term procedurally using pictures when the number of numbers sought or the terms sought are not excessively large or when the terms sought are overly distant from the general terms. Then, using the formula, it was determined.

At the process stage, students coordinate by constructing a pattern from each sequence by figuring the difference between each term or sequence, collecting information from assignments to develop strategies if the terms sought are small and can be solved procedurally while constructing a formula for significant terms. Then, using the difference of succeeding terms in a row, arrange the patterns according to the sequence to calculate the following sequence. Following that, students reverse the process by sequencing/compiling preliminary information related to the problem and figuring out the term in the second row depending on the outcomes of the last term in the first row.

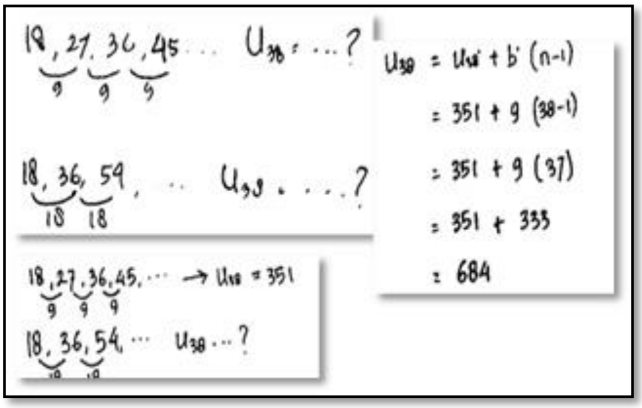


Figure 5. Student Work Results of the Object Stage

At the object stage, based on Figure 5, students show that the pattern has specific properties and characteristics; then, they suggest that to determine the next term, if the term searched is the closest, it can be calculated manually. Meanwhile, the formula is used to find an effective term or one too far from a general term. This can be seen during the interview with the subject, who said that if the difference is 2 times, then the formula becomes $Un = Un' + (n-1)\ b'$, meaning that a is replaced with $U_{n'}$. Then U_{26} is the first row, and the difference used is the difference in the first row or b'.

According to the description above, students encapsulate the process at the object stage by describing and demonstrating that the pattern in the number pattern sequence has specific properties and characteristics; that is, each sequence has a first term and a pattern with the exact differentiator between the two adjacent terms. Additionally, students describe that to find the following term, and a manual computation can be made if the term sought is the closest. Meanwhile, to locate a term that is too huge or too distant from a general term, the formula $U_n = a + (n-1)b$ is applied. Students do de-encapsulation by verifying the patterns they discovered, explicitly finding the first term, the number of terms in the row, and the difference between each term in the row, to avoid making errors or miscalculations when answering questions.

At the schema stage, students demonstrate the pattern's generalizability by associating the action, process, and object of one idea with those of other concepts, particularly a number sequence that can be represented by sequence media and has a regular pattern that forms a number sequence pattern. Students thematized objects by linking the problems that were done previously with the current situation. This was performed by determining the results of the second row based on the first row, still based on the first row, only changing a to Un in the first row and b in the first row to get the results of the second row.

According to the description above, students perform thematization by explaining the generalization properties of the patterns. They also connect a concept's actions, processes, and objects to other ideas. A number sequence can be represented by drawing/image pattern media and has a regular pattern that forms a pattern. In general, a number series is a succession of objects or numbers that have the exact difference between two adjacent terms, consisting of a first term, a second term, and so forth. Meanwhile, to determine the value of the next word in a row, either manually add the difference between the two terms to the preceding term or use the formula approach, namely $U_n = a + (n-1)b$. Students connect previous problems to the current issue, which is to calculate the term in the second row using the results of the first row. They combine the concept of the existing pattern with the sequence, noting that each sequence has a difference between the two consecutive terms. Students find the pattern by calculating the sum of the final term and the difference or by applying the formula to the pattern depicted in the image. Students use current drawing/image patterns to build new thoughts for completing particular sequence patterns. Students connect the concept of an existing pattern to the general sequence by examining the qualities and types of existing patterns, as the principles of all patterns are nearly identical. Because the sole difference between each term is the differentiator, we may deduce the pattern of the sequence formed from the difference between each term. Students find the second row's results based on the first row's results by adding the first row's effects with the differentiator between each word and multiplying by $n-1$.

Discussion

The process of generalizing high performance students' pattern drawing is based on the APOS theory; during the action stage, students work on problem no.1a by finding the value of the next term by adding the differentiator to the last term; students then use the formula method if the sequence has the exact differentiator; surgically, students only use the manual process if the differentiator is different. Students only utilize the formula method to save time on questions with a practical sequence term. Then, in question no. 1b (see Figure 3), students determine the next term using the same procedure as in problem no. 1 part a (see Figure 3). Students utilize the formula $U_n = a + (n-1)b$ in question no. 1, first and second rows of section c because the word in question is too huge, but they check it manually. This is consistent with the view (Dubinsky & Mcdonald, 2001; Maharaj, 2008, 2013) that action is the alteration of an item in response to a stimulus by a human, either explicitly or from memory.

In part an of question no.2 (see Figure 3), the student processes steps to interiorize the action by demonstrating how to manually determine the value of the next term by adding the difference between successive terms and also by using a formula method because the differentiator or discrepancy in the question has a fixed difference or is the same. In part b, students review the procedure for finding the next term, which involves adding the difference of successive terms and applying the formula approach in the preceding problem involving number patterns in the form of numbers. Pupils used the manual method solely when confronted with questions, including several differentiators. Students reflect procedurally on the transformation processes by manually calculating the nth term, adding the differentiator of successive terms, and utilizing the formula $U_n = a + (n-1)b$. This is consistent with the view (Borji & Martínez-Planell, 2020; Dubinsky, 2000; Maharaj, 2013) that individuals interiorize acts by mentally repeating and reflecting on them to visualize and explain the transition without having to perform them openly.

Students coordinate in question number 2 (see Figure 3) throughout the process stage by collecting a pattern from each row, finding the difference between each term, and compiling knowledge about the assignment to build a strategy. If the term being searched is short, manual addition is required; however, manual addition is not required if the term is extended. A formula is required for a sequence with a practical term, and when the differentiators are the same, patterns must be formed to find the next term by finding the differentiator of subsequent terms in the series. Students perform a reversal at this process stage by tracing/compiling preliminary information linked to the problem. If the term is little, students attempt to solve it manually and then recheck using the formula approach; however, if the word is vast, they immediately apply the formula method. This result differs from previous research by Marion et al. (2015), which found that most research subjects used the formula in the book. In other words, the research subjects used procedures to solve number patterns. The difference in the results of this study is because the n as the difference used in this study is relatively small, so they are more flexible and easier to solve manually than using the formula in the book.

Students demonstrate that the pattern has specific traits and characteristics at the object stage, and in the first portion, the difference between successive terms is always the same and fixed. When the differentiator is selected to determine the following term, students relate the challenges they encountered to previously known problems using the formula $U_n = a + (n-1)b$. Students must utilize the manual technique if the sequence contains an unknown differentiator. Students conceptualize the following term by stating that if the term they are looking for is small, it can be calculated manually and then verified with a formula. However, if students are looking for a considerable term with the exact difference/differentiator, they will utilize the procedure$U_n = a + (n-1)b$. Students connect their prior work with the current task, which is to compute the following term in the second row using two times the result from the first row.

Students made a summary during the object stage by outlining the characteristics of each sequence, which include the fact that each sequence has a fixed difference and is a matter of the pattern of the number sequence in the form of a sequence. Students explain the pattern drawing based on recurring differences to obtain the following term and then describe the subsequent term based on the pattern drawing they discovered. Students do de-encapsulation by verifying the patterns drawing discovered, notably finding the first term, the number of terms in the row, and the difference between each term in the row, to ensure that no errors or miscalculations occur during the process. This is consistent with Dubinsky and Mcdonald (2001) view that new processes are produced by accommodating the existing processes; when a current process grows into a technique capable of being modified by an action, then the process is encapsulated into an object.

Students demonstrate the pattern's generalizability at the schema stage by associating one concept's action, process, and object with those of other ideas (Borji et al., 2018; Fuentealba et al., 2019; Sunzuma & Maharaj, 2019). If the differentiator of successive terms is the same or fixed, we can use the formula $U_n = a + (n-1)b$ and the manual method.

At the schema stage, students demonstrate the pattern drawing's generalization properties by associating one concept's action, process, and object with those of other ideas. For example, a number sequence can utilize sequence media and has a regular pattern that forms a number sequence pattern. An object or number with the exact differentiator as the difference between two adjacent words has a first term, a second term, and so forth. Meanwhile, the manual approach for determining the value of a term in a row is to add the difference between the two terms to the previous term and then use the formula method, thus $U_n = a + (n-1)b$. This is consistent with Dubinsky and Mcdonald (2001) definition of a schema as a collection of individual actions from actions, processes, objects, and other schemas connected by some general principles to create a mental framework for dealing with problem situations involving these concepts.

At the schema stage, students thematize objects by relating them to previously solved problems, figuring the following term in the second row based on the results of the previous row, and connecting the existing pattern concept to the sequence in general. Each row pattern contains a differentiator between two consecutive terms. Find the pattern by adding the final term to the difference or applying the formula to the pattern depicted in the image. Students use current sequence patterns as a starting point for developing new concepts for completing specific sequence patterns. Students connect the idea of an existing pattern to the general sequence by examining the qualities and types of existing patterns, as the principles of all sorts of patterns are nearly identical. The only variation is the differentiator in each term; thus, we may deduce the pattern of the series created from the differentiator in each term. Students determine the second row's results based on the first row's results by adding the first row's effects to the differentiator between each word and multiplying by $n - 1$. This is consistent with (Cooley et al., 2007) assertion that humans may execute schema thematization, as evidenced by their capacity to demonstrate the relationship between the schema's concepts and accessible and re-evaluable schema components. According to (Arnon et al., 2014b), thematization is a method that changes a schema into an object, allowing actions or processes on the schema to be performed.

Conclusion

The study's findings indicate that when students are given a sequence, they use a number sequence pattern to calculate the value of the next term. Students interiorize the action throughout the process stage by continuously figuring out the value of the following term and describing how it was determined. Students represent a sequence's differentiator by seeing the pattern of number sequences derived numerous times and reflecting procedurally on the phases of transformation. Students coordinate during the process stage by assembling patterns from each number sequence, collecting information from assignments to develop strategies, and compiling patterns drawing to determine the next term. Students reverse throughout the process stage by retracing their prior knowledge about the problem. Students begin by demonstrating that the number sequence pattern has particular traits and characteristics, explain the relationship between this problem and the previously comprehended problem, then mentally estimate the value of the following term. Students describe the features and patterns of each number sequence before defining the next word in the pattern they discovered. De-encapsulation is accomplished by students evaluating the observed pattern and comparing it to the pattern of the sequence of numbers discovered. Students demonstrate the generalizability of number sequence patterns at the schema stage by relating the activities, processes, and objects of one idea to those of other concepts. Students conduct thematization at the schema stage by associating an existing pattern concept with the whole sequence.

Concerning the results of this study, it was found that students tend to deal with sequence problems. They use sequence patterns that are already known from a problem encountered. This is different from the results of research in general, which states that students usually use sequence formulas to solve the issues they face. In addition, this study contributes to solving problems related to pattern drawing and using representations owned by students in the high performance category.

Recommendations

From this research, the researcher provides recommendations for improving problem-solving skills in mathematical pattern problems based on problem-solving by high performance students'. This improvement can be made by providing teaching tools such as worksheets. Schools can also create similar studies because they will positively contribute to the quality of classroom learning. Learning mathematics by using generalization questions of mathematical patterns that can train students' thinking skills will certainly make students understand the subject matter better. For this reason, teachers can also use patterned mathematical questions to teach their students in class because it directs students to develop thinking when students try to solve patterned mathematical problems. In addition, future researchers with an APOS focus can conduct research related to generalizations for mathematical materials such as geometry, calculus, statistic, or other material.

Limitations

The limitation of this research is that the test instrument used is adapted from mathematical pattern problems developed by previous researchers. In addition, this study was limited to only one school. The next opportunity for researchers is to study in high school or elementary school in the high, medium, and low-performance categories.

Acknowledgments

The authors would like to thank the Institute for Research and Community Service at the Universitas Muhammadiyah Makasar, Universitas PGRI Madiun, Universitas Negeri Jember, and Universitas Sarjanawiyata Tamansiswa, which has facilitated this research. In addition, this research has also been funded by the Directorate of Research, Technology, and Community Service, the Ministry of Education and Culture (DRTPM).

Authorship Contribution Statement

Firdaus: Conceptualization, design, editing/reviewing, supervision, final approval. Murtafiah: Conceptualization, design, analysis/ interpretation, writing, editing/reviewing, final approval. Lukitasari: Editing/reviewing, supervision, final approval. Lestari: Editing/reviewing, supervision, final approval. Ernawati: Interpretation, Editing/reviewing, supervision. Widodo: Editing/reviewing, critical manuscript revision, supervision, final approval.

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